# Multidimensional semicircular limits on the free Wigner chaos

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by Ivan Nourdin[1], Giovanni Peccati[2], and Roland Speicher[3]

**Abstract**: We show that, for sequences of vectors of multiple Wigner integrals with respect to a free Brownian motion, componentwise convergence to semicircular is equivalent to joint convergence. This result extends to the free probability setting some findings by Peccati and Tudor (2005), and represents a multidimensional counterpart of a limit theorem inside the free Wigner chaos established by Kemp, Nourdin, Peccati and Speicher (2011).

**Key words**: Convergence in Distribution; Fourth Moment Condition; Free Brownian Motion; Free Probability; Multidimensional Limit Theorems; Semicircular Law; Wigner Chaos.

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## 1. Introduction

Let $W = \{W_t : t \geq 0\}$ be a one-dimensional standard Brownian motion (living on some probability space $(\Omega, \mathscr{F}, P)$). For every $n \geq 1$ and every real-valued, symmetric and square-integrable function $f \in L^2(\mathbb{R}_+^n)$, we denote by $I^W(f)$ the multiple Wiener-Itô integral of $f$, with respect to $W$. Random variables of this type compose the so-called $n$th *Wiener chaos* associated with $f$. In an infinite-dimensional setting, the concept of Wiener chaos plays the same role as that of the Hermite polynomials for the one-dimensional Gaussian distribution, and represents one of the staples of modern Gaussian analysis (see e.g. [5, 10, 13, 15] for an introduction to these topics).

In recent years, many efforts have been made in order to characterize Central Limit Theorems (CLTs) – that is, limit theorems involving convergence in distribution to a Gaussian element – for random variables living inside a Wiener chaos. The following statement gathers the main findings of [14] (Part 1) and [16] (Part 2), and provides a complete characterization of (both one- and multi-dimensional) CLTs on the Wiener chaos.

**Theorem 1.1** (See [14, 16]). (A) *Let $F_k = I^W(f_k)$, $k \geq 1$, be a sequence of multiple integrals of order $n \geq 2$, such that $E[F_k^2] \to 1$. Then, the following two assertions are equivalent, as $k \to \infty$:* **(i)** *$F_k$ converges in distribution to a standard Gaussian random variable $N \sim \mathscr{N}(0,1)$;* **(ii)** *$E[F_k^4] \to 3 = E[N^4]$.*

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(B) *Let $d \geq 2$ and $n_1, ..., n_d$ be integers, and let $(F_k^{(1)}, ..., F_k^{(d)})$, $k \geq 1$, be a sequence of random vectors such that, for every $i = 1, ..., d$, the random variable $F_k^{(i)}$ lives in the $n_i$th Wiener chaos of $W$. Assume that, as $k \to \infty$ and for every $i, j = 1, ..., d$, $E[F_k^{(i)} F_k^{(j)}] \to c(i, j)$, where $c = \{c(i, j) : i, j = 1, ..., d\}$ is a positive definite symmetric matrix. Then, the following two assertions are equivalent, as $k \to \infty$:* **(i)** *$(F_k^{(1)}, ..., F_k^{(d)})$ converges in distribution to a centered $d$-dimensional Gaussian vector $(N_1, ..., N_d)$ with covariance $c$;* **(ii)** *for every $i = 1, ..., d$, $F_k^{(i)}$ converges in distribution to a centered Gaussian random variable with variance $c(i, i)$.*

Roughly speaking, Part (B) of the previous statement means that, for vectors of random variables living inside some fixed Wiener chaoses, *componentwise convergence to Gaussian always implies joint convergence.* The combination of Part (A) and Part (B) of Theorem 1.1 represents a powerful simplification of the so-called 'method of moments and cumulants' (see e.g. [15, Chapter 11] for a discussion of this point), and has triggered a considerable number of applications, refinements and generalizations, ranging from Stein's method to analysis on homogenous spaces, random matrices and fractional processes – see the survey [9] as well as the forthcoming monograph [10] for details and references.

Now, let $(\mathscr{A}, \varphi)$ be a non-commutative tracial $W^*$-probability space (in particular, $\mathscr{A}$ is a von Neumann algebra and $\varphi$ is a trace – se Section 2.1 for details), and let $S = \{S_t : t \geq 0\}$ be a free Brownian motion defined on it. It is well-known (see e.g. [2]) that, for every $n \geq 1$ and every $f \in L^2(\mathbb{R}_+^n)$, one can define a free multiple stochastic integral with respect to $f$. Such an object is usually denoted by $I^S(f)$. Multiple integrals of order $n$ with respect to $S$ compose the so-called $n$th *Wigner chaos* associated with $S$. Wigner chaoses play a fundamental role in free stochastic analysis – see again [2].

The following theorem, which is the main result of [4], is the exact free analogous of Part (A) of Theorem 1.1. Note that the value 2 coincides with the fourth moment of the standard semicircular distribution $S(0, 1)$.

**Theorem 1.2** (See [4])**.** *Let $n \geq 2$ be an integer, and let $(f_k)_{k \in \mathbb{N}}$ be a sequence of mirror symmetric (see Section 2.2 for definitions) functions in $L^2(\mathbb{R}_+^n)$, each with $\|f_k\|_{L^2(\mathbb{R}_+^n)} = 1$. The following statements are equivalent.*

(1) *The fourth moments of the stochastic integrals $I(f_k)$ converge to 2, that is,*

$$\lim_{k \to \infty} \varphi(I^S(f_k)^4) = 2.$$

(2) *The random variables $I^S(f_k)$ converge in law to the standard semicircular distribution $S(0, 1)$ as $k \to \infty$.*

The aim of this paper is to provide a complete proof of the following Theorem 1.3, which represents a free analogous of Part (B) of Theorem 1.1.

**Theorem 1.3.** *Let $d \geq 2$ and $n_1, \ldots, n_d$ be some fixed integers, and consider a positive definite symmetric matrix $c = \{c(i,j) : i,j = 1, ..., d\}$. Let $(s_1, \ldots, s_d)$ be a semicircular family with covariance $c$ (see Definition 2.10). For each $i = 1, \ldots, d$, we consider a sequence $(f_k^{(i)})_{k \in \mathbb{N}}$ of mirror-symmetric functions in $L^2(\mathbb{R}_+^{n_i})$ such that, for all $i, j = 1, \ldots, d$,*

$$\lim_{k \to \infty} \varphi[I^S(f_k^{(i)}) I^S(f_k^{(j)})] = c(i,j). \tag{1.1}$$

*The following three statements are equivalent as $k \to \infty$.*
*(1) The vector $((I^S(f_k^{(1)}), \ldots, I^S(f_k^{(d)}))$ converges in distribution to $(s_1, \ldots, s_d)$.*
*(2) For each $i = 1, \ldots, d$, the random variable $I^S(f_k^{(i)})$ converges in distribution to $s_i$.*
*(3) For each $i = 1, \ldots, d$,*

$$\lim_{k \to \infty} \varphi[I^S(f_k^{(i)})^4] = 2\, c(i,i)^2.$$

*Remark* 1.4. In the previous statement, the quantity $\varphi[I^S(f_k^{(i)}) I^S(f_k^{(j)})]$ equals $\langle f_k^{(i)}, f_k^{(j)} \rangle_{L^2(\mathbb{R}_+^{n_i})}$ if $n_i = n_j$, and equals 0 if $n_i \neq n_j$. In particular, the limit covariance matrix $c$ is necessarily such that $c(i,j) = 0$ whenever $n_i \neq n_j$.

*Remark* 1.5. Two additional references deal with non-semicircular limit theorems inside the free Wigner chaos. In [11], one can find necessary and sufficient conditions for the convergence towards the so-called Marčenko-Pastur distribution (mirroring analogous findings in the classical setting – see [8]). In [3], conditions are established for the convergence towards the so-called 'tetilla law' (or 'symmetric Poisson distribution' – see also [6]).

Combining the content of Theorem 1.3 with those in [4, 16], we can finally state the following Wiener-Wigner transfer principle, establishing an equivalence between multidimensional limit theorems on the classical and free chaoses.

**Theorem 1.6.** *Let $d \geq 1$ and $n_1, \ldots, n_d$ be some fixed integers, and consider a positive definite symmetric matrix $c = \{c(i,j) : i,j = 1, ..., d\}$. Let $(N_1, \ldots, N_d)$ be a $d$-dimensional Gaussian vector and $(s_1, \ldots, s_d)$ be a semicircular family, both with covariance $c$. For each $i = 1, \ldots, d$, we consider a sequence $(f_k^{(i)})_{k \in \mathbb{N}}$ of fully-symmetric functions (cf. Definition 2.2) in $L^2(\mathbb{R}_+^{n_i})$. Then:*

(1) *For all $i, j = 1, \ldots, d$ and as $k \to \infty$, $\varphi[I^S(f_k^{(i)}) I^S(f_k^{(j)})] \to c(i,j)$ if and only if $E[I^W(f_k^{(i)}) I^W(f_k^{(j)})] \to \sqrt{(n_i)!(n_j)!}\, c(i,j)$.*
(2) *If the asymptotic relations in* (1) *are verified then, as $k \to \infty$,*

$$\big(I^S(f_k^{(1)}), \ldots, I^S(f_k^{(d)})\big) \stackrel{\text{law}}{\to} (s_1, \ldots, s_d)$$

*if and only if*

$$\big(I^W(f_k^{(1)}),\dots,I^W(f_k^{(d)})\big) \overset{\text{law}}{\to} \big(\sqrt{(n_i)!}N_1,\dots,\sqrt{(n_d)!}N_d\big).$$

The remainder of this paper is organized as follows. Section 2 gives concise background and notation for the free probability setting. Theorems 1.3 and 1.6 are then proved in Section 3.

## 2. Relevant definitions and notations

We recall some relevant notions and definitions from free stochastic analysis. For more details, we refer the reader to [2, 4, 7].

2.1. **Free probability, free Brownian motion and stochastic integrals.** In this note, we consider as given a so-called *(tracial) $W^*$ probability space* $(\mathscr{A},\varphi)$, where $\mathscr{A}$ is a von Neumann algebra (with involution $X \mapsto X^*$), and $\varphi : \to \mathbb{C}$ is a *tracial state* (or *trace*). In particular, $\varphi$ is weakly continuous, positive (that is, $\varphi(Y) \geq 0$ whenever $Y$ is a nonnegative element of $\mathscr{A}$), faithful (that is, $\varphi(YY^*) = 0$ implies $Y = 0$, for every $Y \in \mathscr{A}$) and tracial (that is, $\varphi(XY) = \varphi(YX)$, for every $X, Y \in \mathscr{A}$). The self-adjoint elements of $\mathscr{A}$ are referred to as *random variables*. The *law* of a random variable $X$ is the unique Borel measure on $\mathbb{R}$ having the same moments as $X$ (see [7, Proposition 3.13]). For $1 \leq p \leq \infty$, one writes $L^p(\mathscr{A},\varphi)$ to indicate the $L^p$ space obtained as the completion of $\mathscr{A}$ with respect to the norm $\|a\|_p = \tau(|a|^p)^{1/p}$, where $|a| = \sqrt{a^*a}$, and $\|\cdot\|_\infty$ stands for the operator norm.

**Definition 2.1.** Let $\mathscr{A}_1,\dots,\mathscr{A}_n$ be unital subalgebras of $\mathscr{A}$. Let $X_1,\dots,X_m$ be elements chosen from among the $\mathscr{A}_i$'s such that, for $1 \leq j < m$, $X_j$ and $X_{j+1}$ do not come from the same $\mathscr{A}_i$, and such that $\varphi(X_j) = 0$ for each $j$. The subalgebras $\mathscr{A}_1,\dots,\mathscr{A}_n$ are said to be *free* or *freely independent* if, in this circumstance, $\varphi(X_1X_2\cdots X_n) = 0$. Random variables are called freely independent if the unital algebras they generate are freely independent.

**Definition 2.2.** The (centered) *semicircular distribution* (or Wigner law) $S(0,t)$ is the probability distribution

$$S(0,t)(dx) = \frac{1}{2\pi t}\sqrt{4t - x^2}\,dx, \quad |x| \leq 2\sqrt{t}. \tag{2.1}$$

Being symmetric around 0, the odd moments of this distribution are all 0. Simple calculations (see e.g. [7, Lecture 2]) show that the even moments can be expressed in therms of the so-called *Catalan numbers*: for non-negative integers $m$,

$$\int_{-2\sqrt{t}}^{2\sqrt{t}} x^{2m} S(0,t)(dx) = C_m t^m,$$

where $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the $m$th Catalan number. In particular, the second moment (and variance) is $t$ while the fourth moment is $2t^2$.

**Definition 2.3.** A *free Brownian motion* $S$ consists of: (i) a filtration $\{\mathscr{A}_t : t \geq 0\}$ of von Neumann sub-algebras of $\mathscr{A}$ (in particular, $\mathscr{A}_s \subset \mathscr{A}_t$, for $0 \leq s < t$), (ii) a collection $S = \{S_t : t \geq 0\}$ of self-adjoint operators in $\mathscr{A}$ such that: (a) $S_0 = 0$ and $S_t \in \mathscr{A}_t$ for every $t$, (b) for every $t$, $S_t$ has a semicircular distribution with mean zero and variance $t$, and (c) for every $0 \leq u < t$, the increment $S_t - S_u$ is free with respect to $\mathscr{A}_u$, and has a semicircular distribution with mean zero and variance $t - u$.

For the rest of the paper, we consider that the $W^*$-probability space $(\mathscr{A}, \varphi)$ is endowed with a free Brownian motion $S$. For every integer $n \geq 1$, the collection of all operators having the form of a multiple integral $I^S(f)$, $f \in L^2(\mathbb{R}_+^n; \mathbb{C}) = L^2(\mathbb{R}_+^n)$, is defined according to [2, Section 5.3], namely: (a) first define $I^S(f) = (S_{b_1} - S_{a_1}) \cdots (S_{b_n} - S_{a_n})$ for every function $f$ having the form

$$f(t_1, ..., t_n) = \mathbf{1}_{(a_1,b_1)}(t_1) \times \ldots \times \mathbf{1}_{(a_n,b_n)}(t_n), \tag{2.2}$$

where the intervals $(a_i, b_i)$, $i = 1, ..., n$, are pairwise disjoint; (b) extend linearly the definition of $I^S(f)$ to 'simple functions vanishing on diagonals', that is, to functions $f$ that are finite linear combinations of indicators of the type (2.2); (c) exploit the isometric relation

$$\langle I^S(f), I^S(g)\rangle_{L^2(\mathscr{A},\varphi)} = \int_{\mathbb{R}_+^n} f(t_1, \ldots, t_n)\overline{g(t_n, \ldots, t_1)}dt_1 \ldots dt_n, \tag{2.3}$$

where $f, g$ are simple functions vanishing on diagonals, and use a density argument to define $I(f)$ for a general $f \in L^2(\mathbb{R}_+^n)$.

As recalled in the Introduction, for $n \geq 1$, the collection of all random variables of the type $I^S(f)$, $f \in L^2(\mathbb{R}_+^n)$, is called the $n$th *Wigner chaos* associated with $S$. One customarily writes $I^S(a) = a$ for every complex number $a$, that is, the Wigner chaos of order 0 coincides with $\mathbb{C}$. Observe that (2.3) together with the above sketched construction imply that, for every $n, m \geq 0$, and every $f \in L^2(\mathbb{R}_+^n)$, $g \in L^2(\mathbb{R}_+^m)$,

$$\varphi[I^S(f)I^S(g)] = \mathbf{1}_{n=m} \times \int_{\mathbb{R}_+^n} f(t_1, \ldots, t_n)\overline{g(t_n, \ldots, t_1)}dt_1 \ldots dt_n, \tag{2.4}$$

where the right hand side of the previous expression coincides by convention with the inner product in $L^2(\mathbb{R}_+^0) = \mathbb{C}$ whenever $m = n = 0$.

### 2.2. **Mirror Symmetric Functions and Contractions.**

**Definition 2.4.** Let $n$ be a natural number, and let $f$ be a function in $L^2(\mathbb{R}_+^n)$.

1. The *adjoint* of $f$ is the function $f^*(t_1, \ldots, t_n) = \overline{f(t_n, \ldots, t_1)}$.
2. $f$ is called *mirror symmetric* if $f = f^*$, i.e. if

$$f(t_1, \ldots, t_n) = \overline{f(t_n, \ldots, t_1)}$$

for almost all $t_1, \dots, t_n \geq 0$ with respect to the product Lebesgue measure

(3) $f$ is called *fully symmetric* if it is real-valued and, for any permutation $\sigma$ in the symmetric group $\Sigma_n$, $f(t_1, \dots, t_n) = f(t_{\sigma(1)}, \dots, t_{\sigma(n)})$ for almost every $t_1, \dots, t_n \geq 0$ with respect to the product Lebesgue measure.

An operator of the type $I^S(f)$ is self-adjoint if and only if $f$ is mirror symmetric.

**Definition 2.5.** Let $n, m$ be natural numbers, and let $f \in L^2(\mathbb{R}_+^n)$ and $g \in L^2(\mathbb{R}_+^m)$. Let $p \leq \min\{n, m\}$ be a natural number. The *pth contraction* $f \overset{p}{\frown} g$ of $f$ and $g$ is the $L^2(\mathbb{R}_+^{n+m-2p})$ function defined by nested integration of the middle $p$ variables in $f \otimes g$:

$$
\begin{aligned}
&f \overset{p}{\frown} g\,(t_1, \dots, t_{n+m-2p}) \\
&= \int_{\mathbb{R}_+^p} f(t_1, \dots, t_{n-p}, s_1, \dots, s_p) g(s_p, \dots, s_1, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \cdots ds_p.
\end{aligned}
$$

Notice that when $p = 0$, there is no integration, just the products of $f$ and $g$ with disjoint arguments; in other words, $f \overset{0}{\frown} g = f \otimes g$.

2.3. **Non-crossing Partitions.** A *partition* of $[n] = \{1, 2, \dots, n\}$ is (as the name suggests) a collection of mutually disjoint nonempty subsets $B_1, \dots, B_r$ of $[n]$ such that $B_1 \sqcup \cdots \sqcup B_r = [n]$. The subsets are called the *blocks* of the partition. By convention we order the blocks by their least elements; i.e. $\min B_i < \min B_j$ iff $i < j$. If each block consists of two elements, then we call the partition a *pairing*. The set of all partitions on $[n]$ is denoted $\mathscr{P}(n)$, and the subset of all pairings is $\mathscr{P}_2(n)$.

**Definition 2.6.** Let $\pi \in \mathscr{P}(n)$ be a partition of $[n]$. We say $\pi$ has a *crossing* if there are two distinct blocks $B_1, B_2$ in $\pi$ with elements $x_1, y_1 \in B_1$ and $x_2, y_2 \in B_2$ such that $x_1 < x_2 < y_1 < y_2$.

If $\pi \in \mathscr{P}(n)$ has no crossings, it is said to be a *non-crossing partition.* The set of non-crossing partitions of $[n]$ is denoted $NC(n)$. The subset of non-crossing pairings is denoted $NC_2(n)$.

**Definition 2.7.** Let $n_1, \dots, n_r$ be positive integers with $n = n_1 + \cdots + n_r$. The set $[n]$ is then partitioned accordingly as $[n] = B_1 \sqcup \cdots \sqcup B_r$ where $B_1 = \{1, \dots, n_1\}$, $B_2 = \{n_1 + 1, \dots, n_1 + n_2\}$, and so forth through $B_r = \{n_1 + \cdots + n_{r-1} + 1, \dots, n_1 + \cdots + n_r\}$. Denote this partition as $n_1 \otimes \cdots \otimes n_r$.

We say that a pairing $\pi \in \mathscr{P}_2(n)$ *respects* $n_1 \otimes \cdots \otimes n_r$ if no block of $\pi$ contains more than one element from any given block of $n_1 \otimes \cdots \otimes n_r$. The set of such respectful pairings is denoted $\mathscr{P}_2(n_1 \otimes \cdots \otimes n_r)$. The set of non-crossing pairings that respect $n_1 \otimes \cdots \otimes n_r$ is denoted $NC_2(n_1 \otimes \cdots \otimes n_r)$.

**Definition 2.8.** Let $n_1, \ldots, n_r$ be positive integers, and let $\pi \in \mathscr{P}_2(n_1 \otimes \cdots \otimes n_r)$. Let $B_1$, $B_2$ be two blocks in $n_1 \otimes \cdots \otimes n_r$. Say that $\pi$ *links* $B_1$ *and* $B_2$ if there is a block $\{i, j\} \in \pi$ such that $i \in B_1$ and $j \in B_2$.

Define a graph $C_\pi$ whose vertices are the blocks of $n_1 \otimes \cdots \otimes n_r$; $C_\pi$ has an edge between $B_1$ and $B_2$ iff $\pi$ links $B_1$ and $B_2$. Say that $\pi$ *is connected* with respect to $n_1 \otimes \cdots \otimes n_r$ (or that $\pi$ *connects the blocks of* $n_1 \otimes \cdots \otimes n_r$) if the graph $C_\pi$ is connected. We shall denote by $NC_2^c(n_1 \otimes \cdots \otimes n_r)$ the set of all non-crossing pairings that both respect and connect $n_1 \otimes \cdots \otimes n_r$.

**Definition 2.9.** Let $n$ be an even integer, and let $\pi \in \mathscr{P}_2(n)$. Let $f : \mathbb{R}_+^n \to \mathbb{C}$ be measurable. The *pairing integral* of $f$ with respect to $\pi$, denoted $\int_\pi f$, is defined (when it exists) to be the constant

$$\int_\pi f = \int f(t_1, \ldots, t_n) \prod_{\{i,j\} \in \pi} \delta(t_i - t_j)\, dt_1 \cdots dt_n.$$

We finally introduce the notion of a semicircular family (see e.g. [7, Definition 8.15]).

**Definition 2.10.** Let $d \geq 2$ be an integer, and let $c = \{c(i, j) : i, j = 1, ..., d\}$ be a positive definite symmetric matrix. A $d$-dimensional vector $(s_1, ..., s_d)$ of random variables in $\mathscr{A}$ is said to be a *semicircular family with covariance* $c$ if for every $n \geq 1$ and every $(i_1, ..., i_n) \in [d]^n$

$$\varphi(s_{i_1} s_{i_2} \cdots s_{i_n}) = \sum_{\pi \in NC_2(n)} \prod_{\{a,b\} \in \pi} c(i_a, i_b).$$

The previous relation implies in particular that, for every $i = 1, ..., d$, the random variable $s_i$ has the $S(0, c(i, i))$ distribution – see Definition 2.2.

For instance, one can rephrase the defining property of the free Brownian motion $S = \{S_t : t \geq 0\}$ by saying that, for every $t_1 < t_2 < \cdots < t_d$, the vector $(S_{t_1}, S_{t_2} - S_{t_1}, ..., S_{t_d} - S_{t_{d-1}})$ is a semicircular family with a diagonal covariance matrix such that $c(i, i) = t_i - t_{i-1}$ (with $t_0 = 0$), $i = 1, ..., d$.

## 3. Proof of the main results

A crucial ingredient in the proof of Theorem 1.3 is the following statement, showing that contractions control all important pairing integrals. This is the generalization of Proposition 2.2. in [4] to our situation.

**Proposition 3.1.** *Let $d \geq 2$ and $n_1, \ldots, n_d$ be some fixed positive integers. Consider, for each $i = 1, \ldots, d$, sequences of mirror-symmetric functions $(f_k^{(i)})_{k \in \mathbb{N}}$ with $f_k^{(i)} \in L^2(\mathbb{R}_+^{n_i})$, satisfying:*

- *There is a constant $M > 0$ such that $\|f_k^{(i)}\|_{L^2(\mathbb{R}_+^{n_i})} \leq M$ for all $k \in \mathbb{N}$ and all $i = 1, \ldots, d$.*
- *For all $i = 1, \ldots, d$ and all $p = 1, \ldots, n_i - 1$,*

$$\lim_{k \to \infty} f_k^{(i)} \overset{p}{\frown} f_k^{(i)} = 0 \qquad in \quad L^2(\mathbb{R}_+^{2n_i - 2p}).$$

*Let $r \geq 3$, and let $\pi$ be a connected non-crossing pairing that respects $n_{i_1} \otimes \cdots \otimes n_{i_r}$: $\pi \in NC_2^c(n_{i_1} \otimes \cdots \otimes n_{i_r})$. Then*

$$\lim_{k\to\infty} \int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)} = 0.$$

*Proof.* In the same way as in [4] one sees that without restriction (i.e., up to a cyclic rotation and relabeling of the indices) one can assume that

$$\int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)} = \int_{\pi'} (f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}) \otimes (f_k^{(i_3)} \otimes \cdots \otimes f_k^{(i_r)}),$$

where $0 < 2p < n_{i_1} + n_{i_2}$ and

$$\pi' \in NC_2^c\big((n_{i_1} + n_{i_2} - 2p) \otimes n_{i_3} \otimes \cdots \otimes n_{i_r}\big).$$

Note that $0 < 2p < n_{i_1}+n_{i_2}$ says that $f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}$ is not a trivial contraction (trivial means that either nothing or all arguments are contracted); of course, in the case $n_{i_1} \neq n_{i_2}$ it is allowed that $p = \min(n_{i_1}, n_{i_2})$.

By Lemma 2.1. of [4] we have then

$$\begin{aligned} &|\int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)}| \\ &\qquad \leq \|f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}\|_{L^2(\mathbb{R}_+^{n_{i_1}+n_{i_2}-2p})} \cdot \|f_k^{(i_3)}\|_{L^2(\mathbb{R}_+^{n_{i_3}})} \cdots \|f_k^{(i_r)}\|_{L^2(\mathbb{R}_+^{n_{i_r}})} \\ &\qquad \leq \|f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}\|_{L^2(\mathbb{R}_+^{n_{i_1}+n_{i_2}-2p})} \cdot M^{r-2}. \end{aligned}$$

Now we only have to observe that, by also using the mirror symmetry of $f_k^{(i_1)}$ and $f_k^{(i_2)}$, we have

$$\begin{aligned} &\|f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}\|^2_{L^2(\mathbb{R}_+^{n_{i_1}+n_{i_2}-2p})} = \left\langle f_k^{(i_1)} \overset{n_{i_1}-p}{\frown} f_k^{(i_1)}, f_k^{(i_2)} \overset{n_{i_2}-p}{\frown} f_k^{(i_2)} \right\rangle_{L^2(\mathbb{R}_+^{2p})} \\ &\leq \|f_k^{(i_1)} \overset{n_{i_1}-p}{\frown} f_k^{(i_1)}\|_{L^2(\mathbb{R}_+^{2p})} \cdot \|f_k^{(i_2)} \overset{n_{i_2}-p}{\frown} f_k^{(i_2)}\|_{L^2(\mathbb{R}_+^{2p})}. \end{aligned}$$

According to our assumption we have, for each $i = 1, \ldots, d$ and each $q = 1, \ldots, n_i - 1$, that

$$\lim_{k\to\infty} f_k^{(i)} \overset{q}{\frown} f_k^{(i)} = 0 \qquad \text{in} \qquad L^2(\mathbb{R}_+^{2n_i-2q}).$$

Since now at least one of the two contractions $\overset{n_{i_1}-p}{\frown}$ and $\overset{n_{i_2}-p}{\frown}$ is non-trivial, we can choose either $q = n_{i_1} - p$, $i = i_1$ or $q = n_{i_2} - p$, $i = i_2$ in the above, and this implies that

$$\lim_{k\to\infty} \|f_k^{(i_1)} \overset{p}{\frown} f_k^{(i_2)}\|_{L^2(\mathbb{R}_+^{n_{i_1}+n_{i_2}-2p})} = 0,$$

which gives our claim. $\square$

We can now provide a complete proof of Theorem 1.3.

*Proof of Theorem 1.3.* The equivalence between (2) and (3) follows from [4]. Clearly, (1) implies (3), so we only have the prove the reverse implication. So let us assume (3). Note that, by Theorem 1.6 of [4], this is equivalent to the fact that all non-trivial contractions of $f_k^{(i)}$ converge to 0; i.e., for each $i = 1, \dots, d$ and each $q = 1, \dots, n_i - 1$ we have

$$\lim_{k\to\infty} f_k^{(i)} \overset{q}{\frown} f_k^{(i)} = 0 \qquad \text{in} \qquad L^2(\mathbb{R}_+^{2n_i-2q}). \tag{3.1}$$

We will use statement (3) in this form. In order to show (1), we have to show that any moment in the variables $I(f_k^{(1)}), \dots, I(f_k^{(d)})$ converges, as $k \to \infty$, to the corresponding moment in the semicircular variables $s_1, \dots, s_d$. So, for $r \in \mathbb{N}$ and positive integers $i_1, \dots, i_r$, we consider the moments

$$\varphi\left[I^S(f_k^{(i_1)}) \cdots I^S(f_k^{(i_r)})\right].$$

We have to show that they converge, for $k \to \infty$, to the corresponding moment $\varphi(s_{i_1} \cdots s_{i_r})$. Note that our assumption (1.1) says that

$$\lim_{k\to\infty} \varphi[I^S(f_k^{(i)}) I^S(f_k^{(j)})] = c(i,j) = \varphi(s_i s_j).$$

By Proposition 1.38 in [4] we have

$$\varphi\left[I^S(f_k^{(i_1)}) \cdots I^S(f_k^{(i_r)})\right] = \sum_{\pi \in NC_2(n_{i_1} \otimes \cdots \otimes n_{i_r})} \int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)}.$$

By Remark 1.33 in [4], any $\pi \in NC_2(n_{i_1} \otimes \cdots \otimes n_{i_r})$ can be uniquely decomposed into a disjoint union of connected pairings $\pi = \pi_1 \sqcup \cdots \sqcup \pi_m$ with $\pi_q \in NC_2^c(\bigotimes_{j\in I_q} n_{i_j})$, where $\{1, \dots, r\} = I_1 \sqcup \cdots \sqcup I_m$ is a partition of the index set $\{1, \dots, r\}$. The above integral with respect to $\pi$ factors then accordingly into

$$\int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)} = \prod_{q=1}^m \int_{\pi_q} \bigotimes_{j\in I_q} f_k^{(i_j)}.$$

Consider now one of those factors, corresponding to $\pi_q$. Since $\pi_q$ must respect $\bigotimes_{j\in I_q} n_{i_j}$, the number $r_q := \#I_q$ must be strictly greater than 1. On the other hand, if $r_q \geq 3$, then, from (3.1) and Proposition 3.1, it follows that the corresponding pairing integral $\int_{\pi_q} \bigotimes_{j\in I_q} f_k^{(i_j)}$ converges to 0 in $L^2$. Thus, in the limit, only those $\pi$ make a contribution, for which all $r_q$ are equal to 2, i.e., where each of the $\pi_q$ in the decomposition of $\pi$ corresponds to a complete contraction between two of the appearing functions. Let $NC_2^2(n_{i_1} \otimes \cdots \otimes n_{i_r})$ denote the set of those pairings $\pi$. So we get

$$\lim_{k\to\infty} \varphi\left[I(f_k^{(i_1)}) \cdots I(f_k^{(i_r)})\right] = \sum_{\pi \in NC_2^2(n_{i_1} \otimes \cdots \otimes n_{i_r})} \lim_{k\to\infty} \int_\pi f_k^{(i_1)} \otimes \cdots \otimes f_k^{(i_r)},$$

We continue as in [4]: each $\pi \in NC_2^2(n_{i_1} \otimes \cdots \otimes n_{i_r})$ is in bijection with a non-crossing pairing $\sigma \in NC_2(r)$. The contribution of such a $\pi$ is the product of the complete contractions for each pair of the corresponding $\sigma \in NC_2(r)$; but the complete contraction is just the $L^2$ inner product between the paired functions, i.e.,

$$\lim_{k\to\infty} \varphi\left[I^S(f_k^{(i_1)})\cdots I^S(f_k^{(i_r)})\right] = \sum_{\sigma\in NC_2(r)} \prod_{\{s,t\}\in\sigma} c(i_s, i_t).$$

This is exactly the moment $\varphi(s_{i_1}\cdots s_{i_r})$ of a semicircular family $(s_1,\ldots,s_d)$ with covariance matrix $c$, and the proof is concluded. □

We conclude this paper with the proof of Theorem 1.6.

*Proof of Theorem 1.6.* Point (1) is a simple consequence of the Wigner isometry (3.2) (since each $f_k^{(i)}$ is fully symmetric, $f_k^{(i)}$ is in particular mirror-symmetric), together with the classical Wiener isometry which states that

$$E[I^W(f)I^W(g)] = \mathbf{1}_{n=m} \times n!\langle f, g\rangle_{L^2(\mathbb{R}_+^n)} \tag{3.2}$$

for every $n, m \geq 0$, and every $f \in L^2(\mathbb{R}_+^n)$, $g \in L^2(\mathbb{R}_+^m)$. For point (2), we observe first that the case $d = 1$ is already known, as it corresponds to [4, Theorem 1.8]. Consider now the case $d \geq 2$. Let us suppose that $\big(I^S(f_k^{(1)}),\ldots,I^S(f_k^{(d)})\big) \overset{\text{law}}{\to} (s_1,\ldots,s_d)$. In particular, $I^S(f_k^{(i)}) \overset{\text{law}}{\to} s_i$ for all $i = 1,\ldots,d$. By [4, Theorem 1.8] (case $d = 1$), this implies that $I^W(f_k^{(i)}) \overset{\text{law}}{\to} \sqrt{(n_i)!}N_i$. Since the asymptotic relations in (1) are verified, Theorem 1.1**(B)** leads then to $\big(I^W(f_k^{(1)}),\ldots,I^W(f_k^{(d)})\big) \overset{\text{law}}{\to} \big(\sqrt{(n_i)!}N_1,\ldots,\sqrt{(n_d)!}N_d\big)$, which is the desired conclusion. The converse implication follows exactly the same lines,and the proof is concluded. □

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